B.E.(M.D.U.)

First Semester Examination, 2008-09

Mathematics-1 (MATH-1)

Note : Attempt **FIVE** questions in all, selecting two questions from each part.

Part-A

Q. 1. (a) Test the convergence or divergence of the series :

$$\frac{2}{1^p}+\frac{3}{2^p}+\frac{4}{3^p}+\dots\dots\dots .$$

Ans. $\frac{1}{2^p}+\frac{1}{3^p}+\frac{1}{4^p}+\dots\dots\dots\infty$

Here $$u_n=\frac{1}{n^p}=f(n)$$

$\therefore$ $$f(x)=\frac{1}{x^p}$$

For $x>1$, $f(x)$ is positive and monotonic decreasing.

$\therefore$ Cauchy Integral test is applicable.

Case I : When $p\neq 1$

$$\int_1^x f(x)dx=\int_1^\infty \frac{1}{x^p}dx=\int_1^\infty x^{-p}dx=\left[\frac{x^{-p+1}}{-p+1}\right]_1^\infty$$

Subcase (i) When $p>1$, $p-1$ is positive so that,

$$\int_1^x f(x)dx=\frac{-1}{p-1}\left[\frac{1}{x^{p-1}}\right]_1^\infty=\frac{-1}{p-1}[0-1]=\frac{1}{p-1}=\text{finite}$$

$\Rightarrow$ $$\int_1^\infty f(x)dx \text{ coverages} \Rightarrow \Sigma u_n \text{ is convergent}$$

Subcase (ii) : When $0<p<1$, $1-p$ is positive so that,

$$\int_1^\infty f(x)dx=\frac{1}{1-p}\left[x^{1-p}\right]_1^\infty=\frac{1}{1-p}[\infty-1]=\infty$$

$\Rightarrow$ $$\int_1^\infty f(x)dx \text{ diverges} \Rightarrow \Sigma u_n \text{ is divergent}$$

Case II : When $p=1$, $f(x)=\frac{1}{x}$

$$\int_1^\infty f(x)\,dx = \int_1^\infty \frac{1}{x}\,dx = [\log x]_1^\infty = \infty - \log 1 = \infty$$

$$\Rightarrow \qquad \int_1^\infty f(x)\,dx \text{ diverges} \Rightarrow \Sigma u_n \text{ is divergent.}$$

Hence, Σu_n converges if $p > 1$ and diverges if $p \le 1$.

Q. 1. (b) Discuss the convergence of the series :

$$\frac{x}{1} + \frac{1}{2}\cdot\frac{x^3}{3} + \frac{1.3}{2.4}\frac{x^5}{5} + \frac{1.3.5}{2.4.6}\cdot\frac{x^9}{7} + \ldots\ldots(x > 0).$$

Ans. $\frac{x}{1} + \frac{1}{2}\cdot\frac{x^3}{3} + \frac{1.3}{2.4}\frac{x^5}{5} + \frac{1.3.5}{2.4.6}\frac{x^7}{7} + \ldots\ldots x > 0$

Neglecting the first term,

We have, $$u_n = \frac{1.3.5\ldots\ldots(2n-1)}{2.4.6\ldots\ldots(2n)}\frac{x^{2n+1}}{2n+1}$$

And $$u_{n+1} = \frac{1.3.5\ldots\ldots(2n-1)(2n+1)}{2.4.6\ldots\ldots\ldots(2n)(2n+2)}\frac{x^{2n+3}}{2n+3}$$

$$\therefore \qquad \frac{u_n}{u_{n+1}} = \frac{2n+2}{2n+1}\cdot\frac{2n+3}{2n+1}\times\frac{1}{x^2} = \frac{2n\left(1+\frac{1}{n}\right)2n\left(1+\frac{3}{2n}\right)}{2n\left(1+\frac{1}{2n}\right)2n\left(1+\frac{1}{2n}\right)}\frac{1}{x^2}$$

$$= \frac{\left(1+\frac{1}{n}\right)\left(1+\frac{3}{2n}\right)}{\left(1+\frac{1}{2n}\right)^2}\times\frac{1}{x^2}$$

$$\lim_{n\to\infty}\frac{u_n}{u_{n+1}} = \lim_{n\to\infty}\frac{\left(1+\frac{1}{n}\right)\left(1+\frac{3}{2n}\right)}{\left(1+\frac{1}{2n}\right)^2}\times\frac{1}{x^2} = \frac{1}{x^2}$$

$\therefore$ By ratio test, Σu_n is convergent if $\frac{1}{x^2} > 1$ i.e., $x^2 < 1$ and divergent if $\frac{1}{x^2} < 1$ i.e., $x^2 > 1$ if $x^2 = 1$, then ratio test fails.

When, $x^2 = 1$, we have

$$\frac{u_n}{u_{n+1}} = \frac{(2n+2)(2n+3)}{(2n+1)^2} = \frac{4n^2+10n+6}{4n^2+4n+1}$$

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = \lim_{n\to\infty} n\left(\frac{4n^2+10n+6}{4n^2+4n+1} - 1\right)$$

$$= \lim_{n\to\infty} \frac{6n^2+5n}{4n^2+4n+1} = \lim_{n\to\infty} \frac{6+5/4}{4+\frac{4}{n}+\frac{1}{n^2}} = \frac{3}{2}$$

$$\frac{3}{2} > 1$$

$\therefore$ By Rabbe's Test, the series converges.

Hence, Σu_n is convergent if $x^2 \le 1$ and divergent if $x^2 > 1$.

Q. 1. (c) Using the integral test, discuss the convergence of the series :

$$\Sigma \frac{1}{(n+1)^2}.$$

Ans. $\Sigma \frac{1}{(n+1)^2}$

Here, $$u_n = \frac{1}{(n+1)^2} = f(n)$$

$\therefore$ $$f(x) = \frac{1}{(x+1)^2}$$

For $x \ge 1$ $f(x)$ is the and monotonic decreasing. Cauchy's Integral test is applicable.

Now, $$\int_1^\infty f(x)\,dx = \int_1^\infty \frac{dx}{(x+1)^2} = \int_1^\infty (x+1)^{-2}\,dx$$

$$= \left[\frac{(x+1)^{-1}}{-1}\right]_1^\infty$$

$$= \left[\frac{-1}{x+1}\right]_1^\infty$$

$$= \left[0+\frac{1}{2}\right] = \text{finite}$$

$\Rightarrow \int_1^\infty f(x)\,dx$ converges and hence by integral test Σu_n is convergent.

Q. 2. (a) Expand $e^{\sin x}$ by Maclaurin's series up to term containing x^4.

Ans. $f(x) = e^{\sin x}$, $f(0) = 1$

$f'(x) = e^{\sin x}\cos x$, $f'(0) = 1$

$f''(x) = e^{\sin x}(-\sin x) + \cos^2 x e^{\sin x}$

$= e^{\sin x}\left[-\sin x + \cos^2 x\right]$

$= e^{\sin x}\left[-\sin x + 1 - \sin^2 x\right]$

$= e^{\sin}\left[1 - \sin^2 x - \sin x\right]$, $f''(0) = 1$

$f'''(x) = e^{\sin x}[-2\sin x\cos x - \cos x] + \left[1 - \sin^2 x - \sin x\right]e^{\sin x}\cos x$

$= e^{\sin x}\left[-2\sin x\cos x - \cos x + \cos x - \sin^2 x\cos x - \sin x\cos x\right]$

$= e^{\sin x}\left[-3\sin x\cos x - \sin^2 x\cos x\right] = e^{\sin x}\left[-\frac{3}{2}\sin 2x - \sin^2 x\cos x\right]f'''(0) = 0$

$f^{iv}(x) = e^{\sin x}\left[-\frac{3}{2}(\cos 2x)2 - \left[\sin^2 x(-\sin x) + 2\left(\cos^2 x\right)\sin x\right]\right]$

$= e^{\sin x}\left[-3\cos 2x + \sin^3 x - 2\cos^2 x\sin x\right] = f^{iv}(0) = -3$ .....so on

$$f(x) - e^{\sin x} = f(0) + xf'(0) + \frac{x^2}{\lfloor 2}f''(0) + \frac{x^3}{\lfloor 3}f'''(0) + \frac{x^4}{\lfloor 4}f^{iv}(0) + \ldots..$$

$$= 1 + x + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3}(0) + \frac{x^4}{\lfloor 4}(-3)$$

$$\boxed{e^{\sin x} = 1 + x + \frac{x^2}{2} - \frac{x^4}{8} + \ldots\ldots}.$$

Q. 2. (b) Show that the radius of convergence at any point of the cycloid $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$ is $4a\cos\theta/2$.

Ans. $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$

$x' = a(1 + \cos\theta)$, $y' = a(\sin\theta)$

$x'' = a(-\sin\theta)$ $y'' = a\cos\theta$

$$\therefore \quad x'^2 + y'^2 = a^2(1+\cos\theta)^2 + a^2\sin^2\theta$$

$$= a^2\left(1+\cos^2\theta + 2\cos\theta + \sin^2\theta\right)$$

$$= a^2(2+2\cos\theta) = 2a^2(1+\cos\theta) = 2a^2 2\cos^2\frac{\theta}{2} = 4a^2\cos^2\frac{\theta}{2}$$

$$x'y'' - y'x'' = a(1+\cos\theta)a\cos\theta - (a\sin\theta)(-a\sin\theta)$$

$$= a^2\left(\cos\theta + \cos^2\theta + \sin^2\theta\right)$$

$$= a^2(\cos\theta + 1) = a^2 2\cos^2\frac{\theta}{2}$$

Radius of curvature $\delta = \dfrac{\left(x'^2+y'^2\right)^{3/2}}{x'y''-y'x''} = \dfrac{\left(4a^2\cos^2\dfrac{\theta}{2}\right)^{3/2}}{2a^2\cos^2\theta/2}$

$$= \frac{8a^3\cos^3\theta/2}{2a^2\cos^2\theta/2}$$

$$\boxed{4a\cos\theta/2}$$

$$\boxed{\delta = 4a\cos\theta/2}.$$

Q. 2. (c) Find the asymptotes of the curve :

$$\mathbf{x^3 + 3x^2y - 4y^3 - x + y + z = 0}$$

Ans. $x^3 + 3x^2y - 4y^3 - x + y + 3 = 0$ ...(1)

Since the coefficients of x^3 and y^3, the highest degree terms in x and y, are constant there are no asymptotes parallel to the x axis and y axis.

To Find Oblique Asymptotes :

Putting $x = 1$, $y = m$ in third degree terms of equation (1),

$$\phi_3(m) = 1 + 3m - 4m^3,\ \phi_2(m) = 0$$

The slopes of asymptotes are roots of $\phi_3(m) = 0$

$$1 + 3m - 4m^3 = 0$$

$$m = 1,\ -\frac{1}{2},\frac{-1}{2} \qquad \text{for } m = 1$$

Also, $\phi_3'(m) = 3 - 12m^2$

$\therefore$ c is given by $\quad c = -\dfrac{\phi_2(m)}{\phi_3'(m)} = 0$

Corresponding asymptotes is

$$\boxed{y = 1x + 0}$$

For the twice repeated value $m = \frac{-1}{2}$

$$\frac{c^2}{\underline{|2}}\phi_3''(m) + c\phi_2'(m) + \phi_1(m) = 0$$

$$\frac{c^2}{2}(-24m) + c(0) + (-1 + m) = 0$$

$$(-12m)c^2 + m - 1 = 0$$

For $m = \frac{-1}{2} \Rightarrow \left(-12 \times \frac{-1}{2}\right)c^2 - \frac{1}{2} - 1 = 0$

$$6c^2 - \frac{3}{2} = 0 \qquad c^2 = \frac{1}{4}$$

$\Rightarrow$ $$c = \pm\frac{1}{2}$$

Corresponding Asymptotes $y = mx + c$

$$y = -\frac{1}{2}x + \frac{1}{2}, \qquad y = -\frac{1}{2}x - \frac{1}{2}$$

$$2y + x - 1 = 0 \qquad 2y + x + 1 = 0$$

Hence asymptotes are

$$\boxed{y - x = 0, 2y + x - 1 = 0, 2y + x + 1 = 0}.$$

Q. 3. (a) If $u = f(r)$ where $r = \sqrt{x^2 + y^2 + z^2}$ show that:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = f''(r) + \frac{2}{r}f'(r).$$

Ans. If $u = f(r)$, $\qquad r = \sqrt{x^2 + y^2 + z^2} \quad \frac{\partial r}{\partial x} = \frac{x}{r}$

To show, $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = f''(r) + \frac{2}{r}f'(r)$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y}\frac{\partial r}{\partial x} = f'(r)\frac{1}{2}\frac{2x}{\sqrt{x^2 + y^2 + z^2}} = \frac{xf'(r)}{r}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{r\left(xf''(r)\frac{\partial r}{\partial x} + f'(r)\right) - (xf'(r))\frac{\partial r}{\partial x}}{r^2}$$

$$= \frac{xrf''(r)}{r^2\left(\frac{x}{r}\right)} + \frac{rf'(r)}{r^2} - \frac{x^2 f'(r)}{r^3}$$

$$= \frac{x^2 f''(r)}{r^2} + \frac{f'(r)}{r} - \frac{x^2 f'(r)}{r^3} \quad ...(1)$$

By

$$\frac{\partial^2 u}{\partial y^2} = \frac{y^2 f''(r)}{r^2} + \frac{f'(r)}{r} - \frac{y^2 f'(r)}{r^3} \quad ...(2)$$

$$\frac{\partial^2 u}{\partial z^2} = \frac{z^2 f''(r)}{r^2} + \frac{f'(r)}{r} - \frac{z^2 f'(r)}{r^3} \quad ...(3)$$

Adding equations (1) + (2) + (3)

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{f''(r)}{r^2}\left(x^2 + y^2 + z^2\right) + \frac{3f'(r)}{r} - \frac{f'(r)}{r^3}\left(x^2 + y^2 + z^2\right)$$

$$= \frac{f''(r)}{r^2}\left(r^2\right) + \frac{3f'(r)}{r} - \frac{r'(s)}{r^3}\left(r^2\right)$$

$$\boxed{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \Rightarrow f''(r) + \frac{2}{r} f'(r)}.$$ **Hence proved.**

Q. 3. (b) Prove that :

$$\int \frac{\tan^{-1} ax}{x\left(1 + x^2\right)} dx = \frac{\pi}{2} \log(a + 1) \text{ when } a \geq 0.$$

Ans.

$$\int_0^\infty \frac{\tan^{-1} ax}{x\left(1 + x^2\right)} dx \quad ...(i)$$

Let

$$F(a) = \int_0^\infty \frac{\tan^{-1} ax}{x\left(1 + x^2\right)} dx \quad ...(i)$$

Differentiating both sides w.r.t., a

$$F'(a) = \int_0^\infty \frac{\partial}{\partial a}\left[\frac{\tan^{-1} ax}{x(1+x)^2}\right] dx = \int_0^\infty \frac{1}{x\left(1 + x^2\right)} \frac{1}{1 + a^2 x^2} x \, dx$$

$$= \int_0^\infty \frac{dx}{\left(1 + x^2\right)\left(1 + a^2 x^2\right)} = \frac{1}{\left(1 - a^2\right)} \int_0^\infty \left[\frac{1}{1 + x^2} - \frac{1}{1 + a^2 x^2}\right] dx$$

Partial fractions by putting $x^2 = 1$

$$= \frac{1}{1-a^2}\left[\tan^{-1} x\right]_0^\infty \frac{-a^2}{1-a^2}\int_0^\infty \frac{dx}{1+a^2x^2}$$

$$= \frac{1}{1-a^2}\left[\tan^{-1}\infty - \tan^{-1} 0\right] - \frac{a^2}{1-a^2}\frac{1}{a^2}\int_0^\infty \frac{dx}{x^2+1/a^2}$$

$$= \frac{1}{1-a^2}\frac{\pi}{2} - \frac{1}{1-a^2}\times a\left[\tan^{-1}\frac{x}{1/a}\right]_0^\infty$$

$$= \frac{1}{1-a^2}\times\frac{\pi}{2} - \frac{a}{1-a^2}\left[\tan^{-1}\infty - \tan^{-1} 0\right]$$

$$= \frac{1}{1-a^2}\left[\frac{\pi}{2} - \frac{a\pi}{2}\right] = \frac{\pi}{2}\frac{(1-a)}{1-a^2} = \frac{\pi}{2(1+a)}$$

Integrating both sides w.r.t. a,

$$F(a) = \frac{\pi}{2}\log(1+a) + c \qquad \text{...(ii)}$$

From equation (i), when $a = 0$, $F(0) = 0$

$\therefore$ From equation (ii) $0 = \frac{\pi}{2}\log 1 + c = 0 + c \Rightarrow c = 0$

$\therefore$

$$F(a) = \frac{\pi}{2}\log(1+a)$$

$$\boxed{\text{Hence} \int_0^\infty \frac{\tan^{-1} ax}{x\left(1+x^2\right)}dx = \frac{\pi}{2}\log(1+a)}$$

Q. 4. (a) Expand $e^x \log(1+y)$ in powers of x and y upto terms of third degree.

Ans. Here,

$f(x,y) = e^x \log(1+y)$, $\quad f(0,0) = 0$

$f_x(x,y) = e^x \log(1+y)$, $\quad f_x(0,0) = 0$

$f_y(x,y) = \frac{e^x}{1+y}$, $\quad f_y(0,0) = 1$

$f_{xx}(x,y) = e^x \log(1+y)$, $\quad f_{xx}(0,0) = 0$

$f_{xy}(x,y) = \frac{e^x}{1+y}$, $\quad f_{xy}(0,0) = 1$

$f_{yy}(x,y) = -e^x/(1+y)^2$, $\quad f_{yy}(0,0) = -1$

$$f_{xxx}(x,y) = e^x \log(1+y),\ f_{xxx}(0,0) = 0$$

$$f_{xxy}(x,y) = e^x / 1+y,\ f_{xxy}(0,0) = 1$$

$$f_{xyy}(x,y) = -e^x / (1+y)^2,\ f_{xyy}(0,0) = -1$$

$$f_{yyy}(x,y) = 2e^x / (1+y)^3,\ f_{yyy}(0,0) = 2$$

$$\therefore \quad e^x \log(1+y) = f(x,y)$$

$$= f(0,0) + \left[xf_x(0,0) + yf_y(0,0)\right] + \frac{1}{\underline{|2}}\left[x^2 f_{xx}(0,0) + 2xy\, f_{xy}(0,0) + y^2 f_{yy}(0,0)\right]$$

$$+ \frac{1}{\underline{|3}}\left[x^3 f_{xxx}(0,0) + 3x^3 y\, f_{xxy}(0,0) + 3xy^2 f_{xyy}(0,0) + y^3 f_{yyy}(0,0) + ..\right]$$

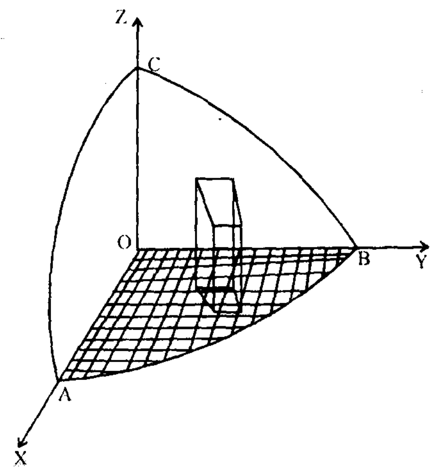
$$= 0 + [x.0 + y.1] + \frac{1}{\underline{|2}}\left[x^2.0 + 2xy.1 + y^2(-1)\right] + \frac{1}{6}\left[x^3.0 + 3x^2y.1 + 3y^2x(-1) + y^3(2)\right] + ...$$

$$= y + xy - \frac{1}{2}y^2 + \frac{1}{2}x^2y - \frac{1}{2}xy^2 + \frac{1}{3}y^2 + ...$$

Q. 4. (b) Find the volume of the greatest rectangular parallelopiped that can be inscribed in the ellipsoid :

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

Ans. An account of symmetry, the required volume is 8 times the volume of the ellipsaid in the positive octant. The volume OABC is the positive octant lies between $z = c\sqrt{\frac{1-x^2}{a^2} - \frac{y^2}{b^2}}$ and the plane xoy and is bound on the sides by the plane x = 0, y = 0.



The ellipsoid cuts the plane XOY in the ellipse,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,\ z = 0$$

$\therefore$ The region OAB above which the volume OABC lies is bounded by, x = 0, x = a and y = 0;

$$y = b\sqrt{\frac{1-x^2}{a^2}}$$

Hence the required volume is,

$$= 8\int_0^a \int_0^{b\sqrt{1-x^2/a^2}} z\,dy\,dx = 8\int_0^a \int_0^{b\sqrt{1-x^2/a^2}} c\sqrt{\frac{1-x^2}{a^2} - \frac{y^2}{b^2}}\,dy\,dx$$

$$= 8\int_0^a\int_0^t c\sqrt{\frac{t^2}{b^2}-\frac{y^2}{b^2}}\,dy\,dx \text{ , where } b\sqrt{\frac{1-x^2}{a^2}} = t$$

$$= 8\int_0^a\int_0^t \frac{c}{b}\sqrt{t^2-y^2}\,dy\,dx$$

$$= 8\int_0^a \frac{c}{b}\left[\frac{y\sqrt{t^2-y^2}}{2}+\frac{t^2}{2}\sin^{-1}\frac{y}{t}\right]_0^t dx$$

$$= \frac{4c}{b}\int_0^a t^2\sin^{-1}1\,dx = \frac{2\pi c}{b}\int_0^c t^2\,dx$$

$$= \frac{2\pi c}{b}\int_0^a b^2\left(1-\frac{x^2}{a^2}\right)dx$$

$$= 2\pi bc\left[x-\frac{x^3}{3a^2}\right]_0^a$$

$$= 2\pi bc\left(a-\frac{a}{3}\right) = \frac{4}{3}\pi abc.$$

Part-B

Q. 5. (a) Find the triple integration, the volume in the positive octant by the co-ordinate planes and the plane $x+2y+3z=4$.

Ans. The given plane is $x+2y+3z=4$

x lies from 0 to $4-2y-3z$

y lies from 0 to $\frac{4-3z}{2}$

z lies from 0 to 4/3

Required volume is,

$$\Rightarrow \int_0^{4/3}\int_0^{\frac{4-3z}{2}}\int_0^{4-2y-3z} dx\,dy\,dz$$

$$\Rightarrow \int_0^{4/3}\int_0^{4-3z/2}[x]_0^{4-2y-3z}\,dy\,dz$$

$$\Rightarrow \int_0^{4/3}\int_0^{(4-3z)/2}(4-2y-3z)\,dy\,dz$$

$$\Rightarrow \int_0^{4/3} \left(4y - y^2 - 3yz\right)_0^{(4-3z)/2} dz$$

$$= \int_0^{4/3} \left[4\left(\frac{4-3z}{2}\right) - \left(\frac{4-3z}{2}\right)^2 - 3z\left(\frac{4-3z}{2}\right)\right] dz$$

$$= \int_0^{4/3} \left[\frac{(8-6z)}{1} - \frac{(16+9z^2-24z)}{4} - \frac{12z-9z^2}{2}\right] dz$$

$$= \int_0^{4/3} \left(\frac{32-24z-16-9z^2+24z-24z+18z^2}{4}\right) dz$$

$$= \frac{1}{4}\int_0^{4/3} \left(9z^2 - 24z + 16\right) dz$$

$$= \frac{1}{4}\left[\frac{9z^3}{3} - \frac{24z^2}{2} + 16z\right]_0^{4/3} = \frac{1}{4}\left[3\left(\frac{4}{3}\right)^3 - 12\left(\frac{4}{3}\right)^2 + 16\left(\frac{4}{3}\right)\right]$$

$$= \frac{1}{4}\left[3 \times \frac{64}{27} - \frac{12 \times 16}{9} + \frac{64}{3}\right]$$

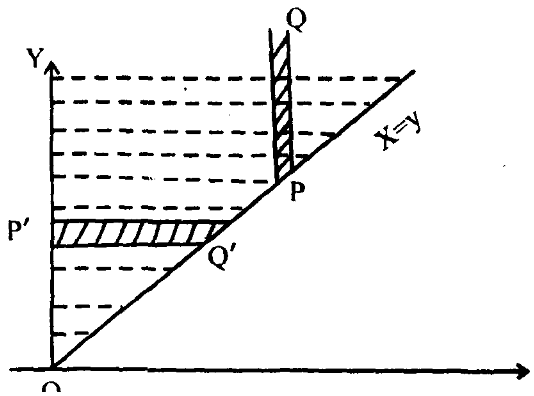
$$\Rightarrow \frac{1}{4}\left[\frac{192-576+576}{27}\right] = \frac{192}{27} \times \frac{1}{4} = \frac{16}{9}.$$

Q. 5. (b) Evaluate by changing the order of integration :

$$\int_0^\infty \int_x^\infty \frac{e^{-y}}{y} dy\, dx.$$

Ans. Here the Integration is first w.r.t. y along the vertical strip PQ under the limits $y = x$ to $y = \infty$ such a strip slides from $x = 0$ to $x = \infty$.

The common region is shown as,



On changing the order of integrate w.r.t x along the horizontal strip P', Q', which extends from $x = 0$ to $x = y$. To cover the region, we then integrate wrt y from $y = 0$ to $y = \infty$.

Thus, $$\int_0^\infty \int_x^\infty \frac{e^{-y}}{y} dy\, dx = \int_0^\infty \int_0^y \frac{e^{-y}}{y} dx\, dy$$

$$= \int_0^\infty \frac{e^{-y}}{y} \, d \ldots \Big]_0 \quad \text{[illegible]}$$

$$= \int_0^\infty y \frac{e^{-y}}{y} dy = \left[\frac{e^{-y}}{-1}\right]_0^\infty = 1.$$

Q. 6. (a) Evaluate the integral by changing into polar-coordinates :

$$\int_0^a \int (x^2 + y^2) \, dy \, dx.$$

Ans. $$\int_0^a \int_0^{\sqrt{a^2-y^2}} (x^2 + y^2) \, dx \, dy$$

Put $$x = r\cos\theta$$
$$y = r\sin\theta$$

So, that $$dx \, dy = r \, d\theta \, dr$$

$$\Rightarrow \int_0^{\pi/2} \int_0^a (r^2) \, r \, dr \, d\theta$$

$$\Rightarrow \int_0^{\pi/2} \int_0^a (r^3) \, dr \, d\theta = \int_0^{\pi/2} \left[\frac{r^4}{4}\right]_0^a d\theta$$

$$= \int_0^{\pi/2} \frac{a^4}{4} d\theta = \frac{a^4}{4} [\theta]_0^{\pi/2}$$

$$= \frac{a^4 \pi}{8}.$$

Q. 6. (b) Express $\int_0^1 x^m (1 - x^n)^p \, dx$ in terms of gamma functions and evaluate

$$\int_0^1 x^5 (1 - x^3)^{10} \, dx.$$

Ans. $$\int_0^1 x^m (1 - x^n)^p \, dx \quad \ldots(1)$$

Let $x^n = t$, $\quad nx^{n-1} dx = dt$

$$\Rightarrow \quad \frac{1}{n}\int_0^1 n x^{m-n+1}\left(1-x^n\right)^p x^{n-1}dx$$

$$= \frac{1}{n}\int_0^1 t^{\frac{m+1-n}{n}}(1-t)^p dt$$

$$= \frac{1}{n}B\left(\frac{m+1-n}{n}+1, p+1\right) \quad \left[\because B(m,n) = \frac{\Gamma m\,\Gamma n}{\Gamma(m+n)}\right]$$

$$= \frac{1}{n}\frac{\Gamma\left(\frac{m+1}{n0}\right)\Gamma p+1}{\Gamma\left(\frac{m+1}{n}+p+1\right)} \qquad \text{...(2)}$$

To deduce $\int_0^1 x^5\left(1-x^3\right)^{10}$ Put m = 5, n = 3, p = 10 in equation (1)

We get, $$= \frac{1}{3}\frac{\Gamma\left(\frac{5+1}{3}\right)\Gamma 11}{\Gamma\left(\frac{5+1}{3}+10+1\right)} = \frac{1}{3}\frac{\Gamma 2\,\Gamma 11}{\Gamma 13}$$

$$= \frac{1}{3}\frac{\lfloor\!\underline{1}\,\lfloor\!\underline{10}}{\lfloor\!\underline{12}} = \frac{1}{12\times 11\times 3}$$

$\because \Gamma n = \lfloor\!\underline{n-1}$

$$= \frac{1}{396}.$$

Q. 7. (a) If $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$, prove that div $\left(r^n\vec{r}\right) = (n+3)r^n$.

Ans. $$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

Prove that $$\text{div}\left(r^n\vec{r}\right) = (n+3)r^n$$

$$\Rightarrow \quad \nabla.\left(r^n\vec{r}\right)$$

$$\Rightarrow \quad \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right).r^n\left(\left(x\hat{i} + y\hat{j} + z\hat{k}\right)\right)$$

$$\Rightarrow \quad \frac{\partial}{\partial x}\left(r^n x\right) + \frac{\partial}{\partial y}\left(r^n y\right) + \frac{\partial}{\partial z}\left(r^n z\right)$$

$$\Rightarrow \left(r^n + xnr^{n-1}\frac{\partial r}{\partial x}\right) + \left(r^n + ynr^{n-1}\frac{\partial r}{\partial y}\right) + \left(r^n + znr^{n-1}\frac{\partial r}{\partial z}\right)$$

We know that, $\frac{\partial r}{\partial x} = \frac{x}{r}$

$$\frac{\partial y}{\partial y} = \frac{y}{r}$$

$$\frac{\partial r}{\partial z} = \frac{z}{r}$$

$$\Rightarrow \left(r^n + xnr^{n-1}\frac{x}{r}\right) + \left(r^n + nyr^{n-1}\frac{y}{r}\right) + \left(r^n + znr^{n-1}\frac{z}{r}\right)$$

$$\Rightarrow \left(r^n + x^2nr^{n-2}\right) + \left(r^n + ny^2r^{n-2}\right) + \left(r^n + nr^{n-2}r^2\right)$$

$$3r^n + nr^{n-2}\left(x^2 + y^2 + z^2\right)$$

$$\Rightarrow 3r^n + nr^{n-2}\left(r^2\right)$$

$$\Rightarrow (3+n)r^n$$

Hence proved.

Q. 7. (b) Show that : curl$\left(\text{curl}\,\vec{V} = \text{grad div.}\vec{V} - \nabla^2\right)$.

Ans.

Let $\vec{V} = V_1\hat{i} + V_2\hat{j} + V_3\hat{k}$

than. $$\text{curl}\,\vec{V} = \nabla \times \vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ V_1 & V_2 & V_3 \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z}\right) + \hat{j}\left(\frac{\partial V_1}{\partial z} - \frac{\partial V_3}{\partial x}\right) + \hat{k}\left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y}\right)$$

$$\therefore \text{curl}\left(\text{curl}\,\vec{V}\right) = \nabla \times \left(\nabla \times \vec{V}\right) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \left(\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z}\right) & \left(\frac{\partial V_1}{\partial z} - \frac{\partial V_3}{\partial x}\right) & \left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y}\right) \end{vmatrix}$$

$$= \Sigma\hat{i}\left\{\frac{\partial}{\partial y}\left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial V_1}{\partial z} - \frac{\partial V_3}{\partial x}\right)\right\}$$

$$= \Sigma\hat{i}\left\{\frac{\partial}{\partial x}\left(\frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z}\right) - \left(\frac{\partial^2 V_1}{\partial y^2} + \frac{\partial^2 V_1}{\partial z^2}\right)\right\}$$

$$= \Sigma\hat{i}\left\{\frac{\partial}{\partial x}\left(\frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z}\right) - \left(\frac{\partial^2 V_1}{\partial x^2} + \frac{\partial^2 V_1}{\partial y^2} + \frac{\partial^2 V_1}{\partial z^2}\right)\right\}$$

$$= \Sigma\hat{i}\left[\frac{\partial}{\partial x}\left(\nabla.\vec{V}\right) - \left(\nabla^2 V_1\right)\right]$$

$$= \Sigma\hat{i}\frac{\partial}{\partial x}\left(\nabla.\vec{V}\right) - \nabla^2 \Sigma i V_1$$

$$= \nabla\left(\nabla.\vec{V}\right) - \nabla^2\vec{V}$$

$= \text{grad}\left(\text{div}\,\vec{V}\right) - \nabla^2 V$. **Hence proved.**

Q. 8. (a) If $\vec{F} = \left(2x^2 - 3z\right)\vec{i} - 2xy\hat{j} - 4x\hat{k}$, then evaluate :

$\iiint \nabla.\vec{F}dV$, where V is bounded by the planes

$x = 0, y = 0, z = 0$ and $2x + 2y + z = 4$.

Ans.

$$\vec{F} = \left(2x^2 - 3z\right)\hat{i} - 2xy\hat{j} + 4x\hat{k}$$

$$\nabla.\vec{F} = \frac{\partial}{\partial x}\left(2x^2 - 3z\right) + \frac{\partial}{\partial y}(-2xy) + \frac{\partial}{\partial z}(-4x)$$

$$= 4x - 2x = 2x$$

$$\therefore \quad \iiint_V \nabla.\vec{F}dv = \iiint_V 2x\,dx\,dy\,dz$$

$$= \int_0^2 \int_0^{2-x} \int_0^{4-2x-2y} 2x\,dz\,dy\,dx$$

$$= \int_0^2 \int_0^{2-x} 2x[z]_0^{4-2x-2y}\,dy\,dz$$

$$= \int_0^2 \int_0^{2-x} 2x(4 - 2x - 2y)dy\,dx$$

$$= \int_0^2 \int_0^{2-x} [4x(2-x) - 4xy]\, dy\, dx$$

$$= \int_0^2 \left[4x(2-x)y - 2xy^2\right]_0^{2-x} dx$$

$$= \int_0^2 \left[4x(2-x)^2 - 2x(2-x)^2\right] dx = \int_0^2 2x(2-x)^2\, dx$$

$$= 2\int_0^2 \left(4x - 4x^2 + x^3\right) dx$$

$$= 2\left[2x^2 - \frac{4x^3}{3} + \frac{x^4}{4}\right]_0^2 = \frac{8}{3}.$$

Q. 8. (b) Verify divergence theorem for :

$$\vec{F} = 4x\hat{i} - 2y^2\hat{j} + z^2\hat{k}$$

taken over the region bounded by the cylinder :

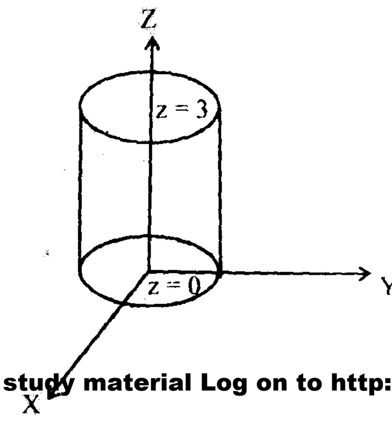
$$x^2 + y^2 = 4, z = 0, z = 3.$$

Ans. $\vec{F} = 4x\hat{i} - 2y^2\hat{j} + z^2\hat{k}$

Since $\operatorname{div} \vec{F} = \frac{\partial}{\partial x}(4x) + \frac{\partial}{\partial y}\left(-2y^2\right) + \frac{\partial}{\partial z}\left(z^2\right) = 4 - 4y + 2z$

$\therefore$ $\iiint \operatorname{div}\vec{F} dv = \iiint (4 - 4y + 2z)\, dx\, dy\, dz$

$$= \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_0^3 (4 - 4y + 2z)\, dz\, dy\, dx$$



$$= \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \left(4z + yz + z^2\right)_0^3 dy\, dx$$

$$= \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (12 - 12y + 9)\, dy\, dz$$

$$= \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} 21 \, dy \, dx$$

Since 21y is and odd function

$$\int_{-a}^{1} 21y \, dy = 0$$

$$= \int_{-2}^{2} 42\sqrt{4-x^2} \, dx$$

$$\Rightarrow \quad 84\int_{0}^{2} \left(4-x^2\right)^{1/2} dx = 84\left[\frac{x\sqrt{4-x^2}}{2} + \frac{4}{2}\sin^{-1}\frac{x}{2}\right]_0^2$$

$$= 84\left[2\sin^{-1} 1\right] = 84\pi \qquad \text{...(1)}$$

To evaluate the surface integral, divide the closed surface S of the cylinder into 3 parts :

S_1 : The circular base in the plane $z = 0$

S_1 : The circular top in the plane $z = 3$

S_3 : The curved surface of the cylinder given by equation $x^2 + y^2 = 4$.

Also, $\iint_S \vec{F}.\hat{n}ds = \iint_{S_1} \vec{F}.\hat{n}ds + \iint_{S_2} \vec{F}.\hat{n}ds + \iiint_S \vec{F}.\hat{n}\,ds$

On $S_1(z=0)$, we have $\hat{n} = -\hat{k}$, $\vec{F} = 4x\hat{i} - 2y^2\hat{j}$

So, that $\vec{F}.\hat{n} = \left(4x\hat{i} - 2y^2\hat{j}\right).\left(-\hat{k}\right) = 0$

$$\therefore \quad \iint_{S_1} \vec{F}.\hat{n}ds = 0$$

On $S_2(z=3)$, we have $\hat{x} = \hat{k}$, $\vec{F} = 4x\hat{i} - 2y^2\hat{j} + 9\hat{k}$

So that, $\vec{F}.n = \left(4x\hat{i} - 2y^2\hat{j} + 9\hat{k}\right).\hat{k} = 9$

$$\therefore \quad \int\int_{S_2} \vec{F}.\hat{n}ds = \int\int_{S_2} 9dx\,dy = 9\int\int_{S_2} dx\,dy = 9 \text{ (Area of surface } S_2\text{)}$$

$$= 9\left(\pi 2^2\right) = 36\pi$$

On S_3, $\quad x^2 + y^2 = 4$

A vector normal to the surface S_3 is given by

$$\nabla\left(x^2+y^2\right)=2x\hat{i}+2y\hat{j}$$

$$\therefore \quad \hat{n}=\frac{2x\hat{i}+2y\hat{j}}{\sqrt{4x^2+4y^2}}=\frac{2x\hat{i}+2y\hat{j}}{\sqrt{4\times 4}}\text{, since } x^2+y^2=4$$

$$\vec{F}.\hat{n}=\left(4x\hat{i}-2y^2\hat{j}+z^2\hat{k}\right).\left(\frac{x\hat{i}+y\hat{j}}{2}\right)=2x^2-y^3$$

Also on S_3 i.e. $x^2+y^2=4$, $x=2\cos\theta$, $y=2\sin\theta$, $ds=2d\theta\,dz$.

To cover the whole surface S_3, z varies from 0 to 3, θ varies from 0 to 2π.

$$\therefore \quad \iint_{S_3} F.n\,ds=\int_0^{2\pi}\int_0^3\left[2(2\cos\theta)^2-(2\sin\theta)^3\right]2dz\,d\theta$$

$$=\int_0^{2\pi}16\left(\cos^2\theta-\sin^3\theta\right)^3 d\theta=48\int_0^{2\pi}\left(\cos^2\theta-\sin^3\theta\right)d\theta=48\pi$$

$$\left[\text{Since}\int_0^{2\pi}\cos^2\theta\,d\theta=2\int_0^{\pi}\cos^2\theta\,d\theta=4\int_0^{\pi/2}\cos^2\theta\,d\theta=4\times\frac{1}{2}\times\frac{\pi}{2}=\pi\int_0^{2\pi}\sin^3\theta d\theta=0\right]$$

$$\therefore \quad \iint_S \vec{F}.\hat{n}ds=0+36\pi+48\pi=84\pi \qquad ...(2)$$

The equality of equations (1) and (2) verifies divergence theorem.